Hints

I provide some detailed derivations for the distributions we have discussed in class. All the following discussions remain within the finite sample setting, i.e. $n < \infty$.

We know that under the classical assumptions,

$$\hat{\boldsymbol{\beta}} \mid \boldsymbol{X} \sim N\left[\boldsymbol{\beta}, \sigma^2 \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\right],$$

therefore under H_0,

$$\frac{c'\hat{\boldsymbol{\beta}} - r}{\sqrt{\sigma^2 c' \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} c}} \sim N(0,1).$$

By replacing σ^2 with sample estimation

$$\hat{\sigma}^2 = \frac{\boldsymbol{e}'\boldsymbol{e}}{n-k-1}$$

then

$$T_n = \frac{c'\hat{\boldsymbol{\beta}} - r}{\sqrt{\hat{\sigma}^2 c' \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} c}} \sim t(n-k-1).$$

To prove this result we need following lemma and theorem from *Multivariate Analysis*.

Lemma 1

(i) If $\boldsymbol{u} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is nonsingular $n \times n$ matrix, then $\boldsymbol{u}'\boldsymbol{\Sigma}\boldsymbol{u} \sim \chi^2(n)$.

(ii) If $\boldsymbol{u} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$ and A is $n \times n$ matrix, then $\boldsymbol{u}'A\boldsymbol{u}/\sigma^2 \sim \chi^2(\text{rank}(A))$.

(iii) If $\boldsymbol{u} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$, A is a $n \times n$ projection matrix, and $A'B = \boldsymbol{0}$, then $\boldsymbol{u}'A\boldsymbol{u}$ and $B'\boldsymbol{u}$ are independent.

*(iv) If $\boldsymbol{u} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$, A and B are both symmetric, then $\boldsymbol{u}'A\boldsymbol{u}$ and $\boldsymbol{u}'B\boldsymbol{u}$ are independent **if and only if** $AB = \boldsymbol{0}$.*

*Besides, if $\boldsymbol{u} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n)$ and A is symmetric, then $\boldsymbol{u}'A\boldsymbol{u}/\sigma^2 \sim \chi^2(\text{rank}(A))$ **if and only if** A is idempotent.*

The following theorem states the sampling distribution of $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$.

Theorem 1

(i) $\frac{(n-k-1)\hat{\sigma}^2}{\sigma^2} \mid \boldsymbol{X} \sim \chi^2(n-k-1)$.

(ii) Conditional $\boldsymbol{X}$, $\hat{\sigma}^2$ *and* $\hat{\boldsymbol{\beta}}$ *are independent.*

Proof.

(i) According to (ii) of **Lemma 1**

$$\frac{(n-k-1)\hat{\sigma}^2}{\sigma^2} \mid \boldsymbol{X} = \frac{\boldsymbol{u}'M\boldsymbol{u}}{\sigma^2} \mid \boldsymbol{X} \sim \chi^2(\text{rank}(M)) = \chi^2(\text{tr}(M)) = \chi^2(n-k-1)$$

(ii) Note that $\hat{\sigma}^2 = \frac{1}{n-k-1}\boldsymbol{u}'M\boldsymbol{u}$ and $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{u}$, then according to (iii) of **Lemma 1** we claim that $\hat{\sigma}^2$ and $\hat{\boldsymbol{\beta}}$ are independent conditional on $\boldsymbol{X}$ because

$$M\left((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\right)' = \mathbf{0}.$$

□

Definition 1 *(**student** t **distribution**) A random variable T follows the student t distribution with q degrees of freedom, written as $T \sim t(q)$, if $T = \frac{U}{\sqrt{V/q}}$ where $U \sim N(0,1)$, $V \sim \chi^2(q)$, and U are V are independent.*

Now we prove that $T_n \sim t(n-k-1)$.

Proof. Let $S_n = (n-k-1)\hat{\sigma}^2/\sigma^2$. Then under $H_0 : c'\boldsymbol{\beta} = r$,

$$\begin{aligned} T_n &= \frac{c'\hat{\boldsymbol{\beta}} - r}{\sqrt{\hat{\sigma}^2 c'(\boldsymbol{X}'\boldsymbol{X})^{-1} c}} = \frac{c'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{\sqrt{\hat{\sigma}^2 c'(\boldsymbol{X}'\boldsymbol{X})^{-1} c}} \\ &= \frac{c'(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sqrt{\sigma^2 c'(\boldsymbol{X}'\boldsymbol{X})^{-1} c}}{\sqrt{\frac{(n-k-1)\hat{\sigma}^2/\sigma^2}{n-k-1}}} \end{aligned}$$

Then the result follows from **Theorem 1** and **Definition 1**. □

Recall the test statistic we construct for hypothesis testing with multiple linear restrictions,

$$F_n \equiv \frac{1}{q}(R\hat{\boldsymbol{\beta}} - r)'\left[\hat{\sigma}^2 R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1}(R\hat{\boldsymbol{\beta}} - r) \sim F(q, n-k-1).$$

Now we prove this result using **Theorem 1** and following definition.

Definition 2 *(F **distribution**) A random variable follows F distribution with (p,q) degrees of freedom, written as $F = \frac{U/p}{V/q}$ where $U \sim \chi^2(p)$, $V \sim \chi^2(q)$, U and V are independent.*

Proof. Since conditional $\boldsymbol{X}$ and under $H_0 : R\boldsymbol{\beta} = r$

$$\hat{\boldsymbol{\beta}} \mid \boldsymbol{X} \sim N\left(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}\right), \quad R\hat{\boldsymbol{\beta}} - r \sim N\left(\mathbf{0}, \sigma^2 R(\boldsymbol{X}'\boldsymbol{X})^{-1}R'\right).$$

Therefore according to (i) of **Lemma 1**, conditional $\boldsymbol{X}$

$$A_n \equiv (R\hat{\boldsymbol{\beta}} - r)'\left[\hat{\sigma}^2 R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1}(R\hat{\boldsymbol{\beta}} - r) \sim F(q, n-k-1) \sim \chi^2(q).$$

Also, we have seen that conditional on $\boldsymbol{X}$

$$S_n \sim \frac{(n-k-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-k-1)$$

and based on **Theorem 1**, S_n and A_n are independent conditional $\boldsymbol{X}$ because $\hat{\sigma}^2$ and $\hat{\boldsymbol{\beta}}$ are independent conditional on $\boldsymbol{X}$. Then the desired result follows by writing

$$F_n = \frac{A_n/q}{S_n/(n-k-1)}$$

conditional on $\boldsymbol{X}$. □