Homework 2

Due Date: May 7 (Wednesday in Class)

1. Textbook page 101 problem 5,6,7,8; page 102 problem 13.

2. A dummy variable takes on only two values: 0 and 1. Define two dummy variables:

$$D_{1i} = \begin{cases} 1 & \text{if the } i\text{-th person is male} \\ 0 & \text{if the } i\text{-th person is female} \end{cases}$$

$$D_{2i} = \begin{cases} 1 & \text{if the } i\text{th person is female} \\ 0 & \text{if the } i\text{th person is male} \end{cases}$$

   Let Y_i be the ith person's wage income, suppose there n_1 men and n_2 women in the sample. Consider following regressions

$$\begin{aligned} Y &= \delta + D_1\alpha_1 + D_2\alpha_2 + u & (1) \\ Y &= D_1\beta_1 + D_2\beta_2 + u & (2) \\ Y &= \mu + D_1\gamma_1 + u & (3) \end{aligned}$$

   (a) Can all the regressions in (1), (2), and (3) be estimated by OLS? Explain your reasoning.

   (b) Explain the relationship between regressions (2) and (3). Is one more general then the other?

   (c) Find the estimator of (1), (2), and (3) if possible. Explain what they are estimating.

3. Consider the model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}$ with the restriction $R\boldsymbol{\beta} = r$, where $\boldsymbol{X}$ is an $n \times (k+1)$ matrix, R is $q \times (k+1)$ matrix with rank $q \leqslant k+1$, and r is a $q \times 1$ vector of constants.

   (a) Explain briefly why the restricted least squares solution $\hat{\boldsymbol{\beta}}_*$ solves the Lagrangian

$$\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\lambda}) = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) + \boldsymbol{\lambda}'(R\boldsymbol{\beta} - r),$$

   where $\boldsymbol{\lambda} \in \mathbf{R}^q$ denotes the Lagrangian multiplier.

(b) Show that

$$\begin{aligned}\hat{\boldsymbol{\beta}}_* &= \hat{\boldsymbol{\beta}} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} (R\hat{\boldsymbol{\beta}} - r), \\ \boldsymbol{\lambda} &= 2\left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} (R\hat{\boldsymbol{\beta}} - r)\end{aligned}$$

where $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$.

(c) Show that if the restriction $R\boldsymbol{\beta} = r$ is true, then

$$\hat{\boldsymbol{\beta}}_* - \boldsymbol{\beta} = \left\{\boldsymbol{I}_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} R\right\} \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{u}.$$

(d) Assume the restriction $R\boldsymbol{\beta} = r$ is true, then under the classical assumptions find the sampling distribution of $\hat{\boldsymbol{\beta}}_* - \boldsymbol{\beta}$.