Solutions to Homework 2

2. (a) No. The regression in (1) cannot be estimated by the OLS whereas the regressions in (2), and (2) can be estimated. For the regression in (1), the regressors form a matrix

$$\boldsymbol{X} = \begin{bmatrix} 1 & D_{11} & D_{21} \\ 1 & D_{12} & D_{22} \\ \vdots & \vdots & \vdots \\ 1 & D_{1n} & D_{2n} \end{bmatrix}$$

which has rank $2 < k = 3$ the number of regressors. This occurs because $D_{1i} + D_{2i} = 1$ for any i This is the case of perfect multicollinearity. For the other two regressions, we don't face this problem and thus they can be estimated by the OLS.

(b) Both regressions (2) and (3) can be used in practice. None is more general than the other.

Note that $\mathrm{E}(Y \mid D_1 = 1, D_2 = 0) = \beta_1$ and $\mathrm{E}(Y \mid D_1 = 0, D_2 = 1) = \beta_2$. They estimate the expected income for the male (β_1) and the expected return for the female β_2 respectively. Note that $\mathrm{E}(Y \mid D_1 = 1) = \gamma_1 + \mu$ and $E(Y \mid D_1 = 0) = \mu$ in regression (3). They estimate the expected income for the male and and the expected income for the female (γ_1) respectively.

So the two regressions are equivalent in that once we know the OLS result from one regression, we can easily recover the OLS result from the other.

(c) We only need to find the estimator of the coefficients in (2) and (3). From the answer to (b), we know that the OLS estimator in regression (2) is given by

$$\hat{\beta}_1 = \bar{Y}_{\text{male}}\ , \hat{\beta}_2 = \bar{Y}_{\text{female}}\ ,$$

where $\bar{Y}_{\text{male}}$ is the average income of the n_1 males and $\bar{Y}_{\text{female}}$ is the average income of the n_2 females. This is because

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{Y} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n_1} Y_i \\ \sum_{i=n_1+1}^{n_1+n_2} Y_i \end{pmatrix} = \begin{pmatrix} \bar{Y}_{\text{male}} \\ \bar{Y}_{\text{female}} \end{pmatrix}.$$

3. (a) This is a constraint minimization problem with q linear constraints given by $R\boldsymbol{\beta} = r$. By the constraint optimization theory, the restricted least squares solution $\hat{\boldsymbol{\beta}}_*$ solves the minimization of the Lagrangian.

(b) By taking the first order derivatives with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$ and set them equal to 0, then

$$\frac{\partial \mathcal{L}(\hat{\boldsymbol{\beta}}_*, \boldsymbol{\lambda})}{\partial \boldsymbol{\beta}} = -2\boldsymbol{X}'\boldsymbol{Y} + 2\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}}_* + R'\boldsymbol{\lambda} = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}(\hat{\boldsymbol{\beta}}_*, \boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}} = R\hat{\boldsymbol{\beta}}_* - r = 0 \tag{2}$$

Rearranging (1) gives

$$\hat{\boldsymbol{\beta}}_* = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{Y} - \frac{1}{2}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\boldsymbol{\lambda} = \hat{\boldsymbol{\beta}} - \frac{1}{2}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\boldsymbol{\lambda} \tag{3}$$

from (2) and (3), we have

$$r = R\hat{\boldsymbol{\beta}}_* = R\hat{\boldsymbol{\beta}} - \frac{1}{2}R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\boldsymbol{\lambda}.$$

Note that $R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'$ is invertible, we have

$$\boldsymbol{\lambda} = 2\left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} (R\hat{\boldsymbol{\beta}} - r) \tag{4}$$

By plugging (4) into (3) we have

$$\hat{\boldsymbol{\beta}}_* = \hat{\boldsymbol{\beta}} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} (R\hat{\boldsymbol{\beta}} - r). \tag{5}$$

(c) Under the null restriction $R\boldsymbol{\beta} = r$

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_* - \boldsymbol{\beta} &= (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} (R\hat{\boldsymbol{\beta}} - R\boldsymbol{\beta}) \\
&= \left\{I_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} R\right\}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \\
&= \left\{I_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\left[R\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R'\right]^{-1} R\right\}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{u}
\end{aligned}$$

(d) Under the null restriction and the classical assumptions, conditional on $\boldsymbol{X}$, $\hat{\boldsymbol{\beta}}_* - \boldsymbol{\beta}$ is a linear combination of multivariate normal distribution ($u \mid X \sim N\left(0, \sigma^2\boldsymbol{X}'\boldsymbol{X}\right)$ here), so

$$\hat{\boldsymbol{\beta}}_* - \boldsymbol{\beta} \mid \boldsymbol{X} \sim N\left(0, \sigma^2\Omega\right)$$

where

$$
\begin{aligned}
\Omega = & \sigma^2 \left\{ I_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \right]^{-1} R \right\} \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{X} \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \\
& \times \left\{ I_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \right]^{-1} R \right\} \\
= & \sigma^2 \left\{ I_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \right]^{-1} R \right\} \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \left\{ I_{k+1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \right]^{-1} R \right\}' \\
= & \sigma^2 \left\{ \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \left[R \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} R' \right]^{-1} R \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \right\}
\end{aligned}
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