时间序列分析：作业 2

1. 考虑 AR(1) 模型 $x_t = \phi x_{t-1} + \varepsilon_t$，$\varepsilon_t \overset{\text{i.i.d.}}{\sim} \left(0, \sigma_\varepsilon^2\right)$，$|\phi| < 1$

   (1) 对任意 i，计算 $\sigma_x^2 = \operatorname{Var}(x_t)$，$\operatorname{Cov}(x_t, x_{t+i})$.

   (2) 通过计算说明 $\hat{\phi}_{ols} = \frac{\frac{1}{T}\sum x_t x_{t-1}}{\frac{1}{T}\sum x_{t-1}^2} \equiv \frac{U_T}{V_T}$，$U_T = \frac{1}{T}\sum x_t x_{t-1}$，$V_T = \frac{1}{T}\sum x_{t-1}^2$.

   (3) Priestley (1981) 有如下公式

$$\begin{aligned}\mathrm{E}\left(x_t x_{t+r} x_s x_{s+r+v}\right) = & \mathrm{E}\left(x_t x_{t+r}\right) \mathrm{E}\left(x_s x_{s+r+v}\right) + \mathrm{E}\left(x_t x_s\right) \mathrm{E}\left(x_{t+r} x_{s+r+v}\right) \\ & + \mathrm{E}\left(x_t x_{s+r+v}\right) \mathrm{E}\left(x_{t+r} x_s\right)\end{aligned}$$

   通过计算说明

$$\operatorname{Cov}\left(x_t^2, x_s x_{s+1}\right) = 2\,\mathrm{E}\left(x_t x_s\right) \mathrm{E}\left(x_t x_{s+1}\right) = 2 \operatorname{Cov}\left(x_t, x_s\right) \operatorname{Cov}\left(x_t x_{s+1}\right)$$

$$\operatorname{Cov}\left(x_t^2, x_s^2\right) = 2\,\mathrm{E}\left(x_t x_s\right) \mathrm{E}\left(x_t x_s\right) = 2\left[\operatorname{Cov}\left(x_t, x_s\right)\right]^2.$$

   (4) 计算 $\operatorname{Cov}(U_T, V_T)$，$\operatorname{Var}(V_T)$.

   (5) 通过计算说明

$$\mathrm{E}\left(\hat{\phi}_{ols}\right) = \mathrm{E}\left(\frac{U_T}{V_T}\right) = \frac{\mathrm{E}\left(U_T\right)}{E\left(V_T\right)} - \frac{\operatorname{Cov}\left(U_T, V_T\right)}{\mathrm{E}^2\left(V_T\right)} + \frac{\mathrm{E}\left(U_T\right) \operatorname{Var}\left(V_T\right)}{\mathrm{E}^3\left(V_T\right)} + o\left(T^{-1}\right).$$

   (6) 在(1)-(5)的基础上通过计算说明

$$\mathrm{E}\left(\hat{\phi}_{ols}\right) = \phi - \frac{2\phi}{T} + o\left(T^{-1}\right).$$

2. 令

$$u_t = \psi(B)\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

   其中，$\varepsilon_t \overset{\text{i.i.d.}}{\sim} (0, \sigma^2)$，$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$，且 $\sum_{j=0}^{\infty} j|\psi_j| < \infty$.

   (1) 说明 $\{u_t\}$ 是宽平稳序列.

   (2) 通过计算证明

$$\begin{aligned}\sum_{i=1}^{t} u_i &= \sum_{i=1}^{t}\sum_{j=0}^{\infty} \psi_j \varepsilon_{i-j} \\ &= \sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{t} \varepsilon_i + \eta_t - \eta_0\end{aligned}$$

   其中，

$$\eta_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}, \quad \alpha_j = -\left(\psi_{j+1} + \psi_{j+2} + \ldots\right)$$

   并说明 $\{\eta_t\}$ 是宽平稳序列.

   (3) 假设 y_t 由 ARIMA$(p,1,q)$ 生成，且 $u_t = \nabla y_t$，利用 (1) 和 (2) 中的结果说明 y_t 可以分解成 ARIMA(0,1,0) 序列和平稳序列之和.