时间序列分析：作业 2

1. 考虑 AR(1) 模型 $x_t = \phi x_{t-1} + \varepsilon_t$，$\varepsilon_t \overset{\text{i.i.d.}}{\sim} \left(0, \sigma_\varepsilon^2\right)$，$|\phi| < 1$

   (1) 对任意 i，计算 $\sigma_x^2 = \mathrm{Var}(x_t)$，$\mathrm{Cov}\left(x_t, x_{t+i}\right)$.

   参考答案

$$\sigma_x^2 = \mathrm{Var}(x_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2}, \quad \mathrm{Cov}\left(x_t, x_{t+i}\right) = \phi^{|i|}\sigma_x^2$$

   (2) 通过计算说明 $\hat{\phi}_{ols} = \frac{\frac{1}{T}\sum x_t x_{t-1}}{\frac{1}{T}\sum x_{t-1}^2} \equiv \frac{U_T}{V_T}$，$U_T = \frac{1}{T}\sum x_t x_{t-1}$，$V_T = \frac{1}{T}\sum x_{t-1}^2$.

   (3) Priestley (1981) 有如下公式

$$\begin{aligned} \mathrm{E}\left(x_t x_{t+r} x_s x_{s+r+v}\right) &= \mathrm{E}\left(x_t x_{t+r}\right)\mathrm{E}\left(x_s x_{s+r+v}\right) + \mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_{t+r} x_{s+r+v}\right) \\ &\quad + \mathrm{E}\left(x_t x_{s+r+v}\right)\mathrm{E}\left(x_{t+r} x_s\right) \end{aligned}$$

   通过计算说明

$$\begin{aligned} \mathrm{Cov}\left(x_t^2, x_s x_{s+1}\right) &= 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_{s+1}\right) = 2\,\mathrm{Cov}\left(x_t, x_s\right)\mathrm{Cov}\left(x_t, x_{s+1}\right) \\ \mathrm{Cov}\left(x_t^2, x_s^2\right) &= 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_s\right) = 2\left[\mathrm{Cov}\left(x_t, x_s\right)\right]^2. \end{aligned}$$

   参考答案

$$\begin{aligned} \mathrm{Cov}\left(x_t^2, x_s x_{s+1}\right) &= \mathrm{E}\left(x_t^2 x_s x_{s+1}\right) - \mathrm{E}\left(x_t^2\right)\mathrm{E}\left(x_s x_{s+1}\right) \\ &= \mathrm{E}\left(x_t^2\right)\mathrm{E}\left(x_s x_{s+1}\right) + \mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_{s+1}\right) + \mathrm{E}\left(x_t x_{s+1}\right)\mathrm{E}\left(x_t x_s\right) - \mathrm{E}\left(x_t^2\right)\mathrm{E}\left(x_s x_{s+1}\right) \\ &= 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_{s+1}\right) = 2\,\mathrm{Cov}\left(x_t, x_s\right)\mathrm{Cov}\left(x_t, x_{s+1}\right) \end{aligned}$$

   类似地可以说明

$$\mathrm{Cov}\left(x_t^2, x_s^2\right) = 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_s\right) = 2\left[\mathrm{Cov}\left(x_t, x_s\right)\right]^2$$

   (4) 计算 $\mathrm{Cov}\left(U_T, V_T\right)$，$\mathrm{Var}\left(V_T\right)$.

   参考答案

$$\begin{aligned} \mathrm{Cov}\left(U_T, V_T\right) &= \mathrm{Cov}\left(\frac{1}{T}\sum x_t x_{t-1}, \frac{1}{T}\sum x_{t-1}^2\right) \\ &= \frac{1}{T^2}\sum\sum \mathrm{Cov}\left(x_t^2, x_s x_{s+1}\right) \\ &= \frac{2}{T^2}\sum\sum \mathrm{Cov}\left(x_t, x_s\right)\mathrm{Cov}\left(x_t, x_{s+1}\right) \\ &= \sigma_x^4 \frac{2}{T^2}\sum\sum \phi^{|t-s|}\phi^{|t-s-1|} \end{aligned}$$

其中，

$$
\begin{aligned}
\frac{2\sigma_x^4}{T^2}\sum\sum\phi^{|t-s|}\phi^{|t-s-1|} &= \frac{2\sigma_x^4}{T^2}\left[T\phi+(T-1)(\phi+\phi^3)+(T-2)(\phi^3+\phi^5)+\cdots+(\phi^{2T-3}+\phi^{2T-1})\right] \\
&= \frac{2\sigma_x^4}{T^2}\left[(2T-1)\phi+(2T-3)\phi^3+(2T-5)\phi^5+\cdots+(2T-3)\phi^{2T-3}+\phi^{2T-1}\right] \\
&= \frac{2\sigma_x^4}{T^2}\frac{2T\phi}{1-\phi^2}+o\left(T^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(V_T\right) &= \mathrm{Var}\left(\frac{1}{T}\sum x_{t-1}^2\right) \\
&= \frac{1}{T^2}\sum\sum\mathrm{Cov}\left(x_t^2,x_s^2\right) \\
&= \frac{2}{T^2}\sum\sum\left[\mathrm{Cov}\left(x_t,x_s\right)\right]^2 \\
&= \frac{2\sigma_x^4}{T^2}\sum\sum\phi^{2|t-s|}
\end{aligned}
$$

其中，

$$
\begin{aligned}
\frac{2\sigma_x^4}{T^2}\sum\sum\phi^{2|t-s|} &= \frac{2\sigma_x^4}{T^2}\left[T+2(T-1)\phi+2(T-2)\phi^2+\cdots+2\phi^{T-1}\right] \\
&= \frac{2\sigma_x^4}{T^2}\frac{T(1+\phi^2)}{1-\phi^2}+o\left(T^{-1}\right)
\end{aligned}
$$

(5) 通过计算说明

$$
\mathrm{E}\left(\hat{\phi}_{ols}\right)=\mathrm{E}\left(\frac{U_T}{V_T}\right)=\frac{\mathrm{E}\left(U_T\right)}{\mathrm{E}\left(V_T\right)}-\frac{\mathrm{Cov}\left(U_T,V_T\right)}{\mathrm{E}^2\left(V_T\right)}+\frac{\mathrm{E}\left(U_T\right)\mathrm{Var}\left(V_T\right)}{\mathrm{E}^3\left(V_T\right)}+o\left(T^{-1}\right).
$$

(6) 在(1)-(5) 的基础上通过计算说明

$$
\mathrm{E}\left(\hat{\phi}_{ols}\right)=\phi-\frac{2\phi}{T}+o\left(T^{-1}\right).
$$

参考答案.

$$
\begin{aligned}
\mathrm{E}\left(\hat{\phi}_{ols}\right)=\mathrm{E}\left(\frac{U_T}{V_T}\right) &= \frac{\mathrm{E}\left(U_T\right)}{\mathrm{E}\left(V_T\right)}-\frac{\mathrm{Cov}\left(U_T,V_T\right)}{\mathrm{E}^2\left(V_T\right)}+\frac{\mathrm{E}\left(U_T\right)\mathrm{Var}\left(V_T\right)}{\mathrm{E}^3\left(V_T\right)}+o\left(T^{-1}\right) \\
&= \frac{\phi\sigma_x^2}{\sigma_x^2}-\frac{2\sigma_x^4}{T^2}\frac{2T\phi}{1-\phi^2}\frac{1}{\sigma_x^4}+\frac{\phi\sigma_x^2 2\sigma_x^4 T(1+\phi^2)}{T^2(1-\phi^2)\sigma_x^6}+o\left(T^{-1}\right) \\
&= \phi-\frac{4\phi}{T(1-\phi^2)}+\frac{2\phi(1+\phi^2)}{T(1-\phi^2)}+o\left(T^{-1}\right) \\
&= \phi+\frac{2\phi\left(1+\phi^2-2\right)}{T(1-\phi^2)}+o\left(T^{-1}\right) \\
&= \phi-\frac{2\phi}{T}+o\left(T^{-1}\right)
\end{aligned}
$$

2. 令

$$
u_t=\psi(B)\varepsilon_t=\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}
$$

其中，$\varepsilon_t \overset{\text{i.i.d.}}{\sim} (0,\sigma^2)$，$\psi(B)=\sum_{j=0}^{\infty}\psi_j B^j$，且 $\sum_{j=0}^{\infty} j|\psi_j|<\infty$.

(1) 说明 $\{u_t\}$ 是宽平稳序列.

参考答案

由 MA(∞) 的形式可知，说明 $\{u_t\}$ 的平稳性只需要证明

$$\mathrm{Var}\,(u_t) = \sum_{j=0}^{\infty} |\psi_j|^2 < \infty$$

而显然 $\sum_{j=0}^{\infty}|\psi_j|^2 < \left(\sum_{j=0}^{\infty}|\psi_j|\right)\left(\sum_{j=0}^{\infty}|\psi_j|\right)$，$\sum_{j=0}^{\infty}|\psi_j| < \sum_{j=0}^{\infty} j|\psi_j| < \infty$，所以 $\sum_{j=0}^{\infty}|\psi_j|^2 < \infty$

(2) 通过计算证明

$$\begin{aligned}\sum_{i=1}^{t} u_i &= \sum_{i=1}^{t}\sum_{j=0}^{\infty}\psi_j\varepsilon_{i-j} \\ &= \sum_{j=0}^{\infty}\psi_j\sum_{i=1}^{t}\varepsilon_i + \eta_t - \eta_0\end{aligned}$$

其中，

$$\eta_t = \sum_{j=0}^{\infty}\alpha_j\varepsilon_{t-j}, \quad \alpha_j = -\left(\psi_{j+1} + \psi_{j+2} + \ldots\right)$$

并说明 $\{\eta_t\}$ 是宽平稳序列.

参考答案

$$\begin{aligned}\sum_{i=1}^{t} u_i &= \sum_{i=1}^{t}\sum_{j=0}^{\infty}\psi_j\varepsilon_{i-j} \\ &= \{\psi_0\varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \cdots + \psi_t\varepsilon_0 + \psi_{t+1}\varepsilon_{-1} + \cdots\} \\ &\quad + \{\psi_0\varepsilon_{t-1} + \psi_1\varepsilon_{t-2} + \psi_2\varepsilon_{t-3} + \cdots + \psi_{t-1}\varepsilon_0 + \psi_t\varepsilon_{-1} + \cdots\} \\ &\quad + \{\psi_0\varepsilon_{t-2} + \psi_1\varepsilon_{t-3} + \psi_2\varepsilon_{t-4} + \cdots + \psi_{t-2}\varepsilon_0 + \psi_{t-1}\varepsilon_{-1} + \cdots\} \\ &\quad + \cdots + \{\psi_0\varepsilon_1 + \psi_1\varepsilon_0 + \psi_2\varepsilon_{-1} + \cdots\} \\ &= \psi_0\varepsilon_t + (\psi_0 + \psi_1)\,\varepsilon_{t-1} + (\psi_0 + \psi_1 + \psi_2)\,\varepsilon_{t-2} + \cdots \\ &\quad + (\psi_0 + \psi_1 + \psi_2 + \cdots + \psi_{t-1})\,\varepsilon_1 + (\psi_1 + \psi_2 + \cdots + \psi_t)\,\varepsilon_0 \\ &\quad + (\psi_2 + \psi_3 + \cdots + \psi_{t+1})\,\varepsilon_{-1} + \cdots \\ &= (\psi_0 + \psi_1 + \psi_2 + \cdots)\,\varepsilon_t - (\psi_1 + \psi_2 + \psi_3 + \cdots)\,\varepsilon_t \\ &\quad + (\psi_0 + \psi_1 + \psi_2 + \cdots)\,\varepsilon_{t-1} - (\psi_2 + \psi_3 + \cdots)\,\varepsilon_{t-1} \\ &\quad + (\psi_0 + \psi_1 + \psi_2 + \cdots)\,\varepsilon_{t-2} - (\psi_3 + \psi_4 + \cdots)\,\varepsilon_{t-2} + \cdots \\ &\quad + (\psi_0 + \psi_1 + \psi_2 + \cdots)\,\varepsilon_1 - (\psi_t + \psi_{t+1} + \cdots)\,\varepsilon_1 \\ &\quad + (\psi_1 + \psi_2 + \psi_3 + \cdots)\,\varepsilon_0 - (\psi_{t+1} + \psi_{t+2} + \cdots)\,\varepsilon_0 \\ &\quad + (\psi_2 + \psi_3 + \psi_4 + \cdots)\,\varepsilon_{-1} - (\psi_{t+2} + \psi_{t+3} + \cdots)\,\varepsilon_{-1} + \cdots\end{aligned}$$

即 $\sum_{i=1}^{t} u_i = \sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{t} \varepsilon_i + \eta_t - \eta_0$，

$$
\begin{aligned}
\eta_t &= \underbrace{-\left(\psi_1+\psi_2+\psi_3+\cdots\right)}_{\alpha_0}\varepsilon_t \underbrace{-\left(\psi_2+\psi_3+\psi_4+\cdots\right)}_{\alpha_1}\varepsilon_{t-1} \\
&\quad \underbrace{-\left(\psi_3+\psi_4+\psi_5+\cdots\right)}_{\alpha_2}\varepsilon_{t-2} - \cdots \\
\eta_0 &= \underbrace{-\left(\psi_1+\psi_2+\psi_3+\cdots\right)}_{\alpha_0}\varepsilon_0 \underbrace{-\left(\psi_2+\psi_3+\psi_4+\cdots\right)}_{\alpha_1}\varepsilon_{-1} \\
&\quad \underbrace{-\left(\psi_3+\psi_4+\psi_3+\cdots\right)}_{\alpha_2}\varepsilon_{-2} - \cdots .
\end{aligned}
$$

此外，

$$
\begin{aligned}
\sum_{j=1}^{\infty}\left|\alpha_j\right| &= |\psi_1+\psi_2+\psi_3+\cdots| + |\psi_2+\psi_3+\psi_4+\cdots| + |\psi_3+\psi_4+\psi_5+\cdots| + \cdots \\
&\leqslant \{|\psi_1|+|\psi_2|+|\psi_3|+\cdots\} + \{|\psi_2|+|\psi_3|+|\psi_4|+\cdots\} + \{|\psi_3|+|\psi_4|+|\psi_s|+\cdots\} + \cdots \\
&= |\psi_1| + 2\,|\psi_2| + 3\,|\psi_3| + \cdots \\
&= \sum_{j=0}^{\infty} j|\psi_j| < \infty
\end{aligned}
$$

所以，$\eta_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$ 是宽平稳序列.

(3) 假设 y_t 由 ARIMA$(p,1,q)$ 生成，且 $u_t = \nabla y_t$，利用 (1) 和 (2) 中的结果说明 y_t 可以分解成 ARIMA(0,1,0) 序列和平稳序列之和.

参考答案

$$
y_t = \sum_{i=1}^{t} \nabla y_t + y_0 = \sum_{i=1}^{t} u_t + y_0 = \sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{t} \varepsilon_t + \eta_t + (Y_0 - \eta_0),
$$

其中，$\sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{t} \varepsilon_t$ 是 ARIMA$(0,1,0)$ 序列，η_t 由 (2) 可知是平稳序列. 这一分解也称之为 Beveridge-Nelson 分解.