时间序列分析：作业 1

1. 考虑随机变量 $\{X_n\}$，且 $\mathbb{P}\left(X_n=\sqrt{n}\right)=\frac{1}{n}$，$\mathbb{P}\left(X_n=0\right)=1-\frac{1}{n}$，说明 $\{X_n\}$ 是否依概率收敛？

   参考答案.
   $\forall\, \varepsilon>0$，
   $$\mathbb{P}\left(|X_n|>\varepsilon\right)=\mathbb{P}\left(X_n=\sqrt{n}\right)=\frac{1}{n}\to 0.$$
   所以，$\{X_n\}$ 依概率收敛到 0。 □

2. 考虑线性回归模型 $y=X\beta+u$，其中
   $$X=\begin{pmatrix}1 & 1\\ 1 & 2\\ \vdots & \vdots\\ 1 & n\end{pmatrix}$$
   且 u 为随机误差向量满足 $\mathbb{E}\left(u\mid X\right)=0$ 且 $\mathrm{Var}\left(u\mid X\right)=\sigma^2 I_n$。显然，该线性回归模型可以等价的表示为 $y_t=\beta_1+\beta_2 t+u_t$，其中 $\beta=(\beta_1,\beta_2)'$，$t=1,\ldots,n$。
   (1) 说明 $\frac{1}{n}X^\top X$ 是否收敛到一个正定矩阵（positive definite matrix）？
   (2) 说明 $X^\top X\to 0$？这个收敛性可以用于说明 β 的最小二乘估计量（OLS）的什么性质？请通过计算说明。

   参考答案.

   (1)
   $$\begin{aligned}\frac{1}{n}X'X=\frac{1}{n}\begin{pmatrix} n & \sum_{t=1}^n t\\ \sum_{t=1}^n t & \sum_{t=1}^n t^2\end{pmatrix}&=\frac{1}{n}\begin{pmatrix} n & \frac{n(n+1)}{2}\\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6}\end{pmatrix}\\ &=\begin{pmatrix}1 & \frac{n+1}{2}\\ \frac{n+1}{2} & \frac{(n+1)(2n+1)}{6}\end{pmatrix},\end{aligned}$$
   因此，不收敛。

   (2)
   $$\begin{aligned}(X'X)^{-1}&=\begin{pmatrix} n & \frac{n(n+1)}{2}\\ \frac{n(n+1)}{2} & \frac{n(n+1)(2n+1)}{6}\end{pmatrix}^{-1}\\ &=\frac{1}{\frac{n^2(n+1)(2n+1)}{6}-\frac{n^2(n+1)^2}{4}}\begin{pmatrix}\frac{n(n+1)(2n+1)}{6} & -\frac{n(n+1)}{2}\\ -\frac{n(n+1)}{2} & n\end{pmatrix}\\ &=\begin{pmatrix} O\left(n^{-1}\right) & O\left(n^{-2}\right)\\ O\left(n^{-2}\right) & O\left(n^{-3}\right)\end{pmatrix}\to 0.\end{aligned}$$

由此，我们可以说明 β 的最小二乘估计量 $\hat{\beta}$ 是 β 的一致估计量，因为 $\forall\, \epsilon > 0$，

$$\begin{aligned}
\mathbb{P}(\|\hat{\beta}-\beta\| \geq \epsilon) &\leq \frac{1}{\epsilon^2}\mathbb{E}\left\{\|\hat{\beta}-\beta\|^2\right\} = \frac{1}{\epsilon^2}\mathbb{E}\left[(\hat{\beta}-\beta)'(\hat{\beta}-\beta)\right] \\
&= \frac{1}{\epsilon^2}\mathbb{E}\,\mathrm{tr}\left[u'X\left(X'X\right)^{-1}\left(X'X\right)^{-1}X'u \mid X\right] \\
&= \frac{1}{\epsilon^2}\,\mathrm{tr}\,\mathbb{E}\left[X\left(X'X\right)^{-1}\left(X'X\right)^{-1}X'\mathbb{E}\left(uu' \mid X\right)\right] \\
&= \frac{\sigma^2}{\epsilon^2}\,\mathrm{tr}\,\mathbb{E}\left[X\left(X'X\right)^{-1}\left(X'X\right)^{-1}X'\right] \\
&= \frac{\sigma^2}{\epsilon^2}\,\mathrm{tr}\left[\left(X'X\right)^{-1}\right] \to 0 \text{ as } \left(X'X\right)^{-1} \to 0.
\end{aligned}$$

□

3. 对于中心化的 ARMA(p,q) 模型 $\Phi(B)x_t = \Theta(B)\varepsilon_t$，$\Phi(B) = 1-\phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$，$1-\theta_1 B - \cdots - \theta_q B^q$，$\varepsilon_t \overset{\text{i.i.d.}}{\sim} \left(0, \sigma_\varepsilon^2\right)$，证明该 ARMA$(p,q)$ 模型可以表示成 $x_t + \sum_{i=1}^{\infty} I_i x_{t-i}$，并以递归形式给出 $\{I_i\}_{i=0}^{\infty}$ 的表达式。

参考答案.
令 $\Theta^{-1}(B)\Phi(B) = \sum_{i=0}^{\infty} I_i B^i$，并注意到 $\Theta^{-1}(B)$ 的表达式，带入并通过待定系数法比较系数给出 $\{I_i\}_{i=0}^{\infty}$ 的表达式。

□

4. 考虑 AR(1) 模型 $x_t = \phi x_{t-1} + \varepsilon_t$，$\varepsilon_t \overset{\text{i.i.d.}}{\sim} \left(0, \sigma_\varepsilon^2\right)$，$|\phi| < 1$

   (1) 对任意 i，计算 $\sigma_x^2 = \mathrm{Var}(x_t)$，$\mathrm{Cov}\left(x_t, x_{t+i}\right)$.

   参考答案

$$\sigma_x^2 = \mathrm{Var}(x_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2}, \quad \mathrm{Cov}\left(x_t, x_{t+i}\right) = \phi^{|i|}\sigma_x^2$$

   (2) 通过计算说明 $\hat{\phi}_{ols} = \frac{\frac{1}{T}\sum x_t x_{t-1}}{\frac{1}{T}\sum x_{t-1}^2} \equiv \frac{U_T}{V_T}$，$U_T = \frac{1}{T}\sum x_t x_{t-1}$，$V_T = \frac{1}{T}\sum x_{t-1}^2$.

   (3) Priestley (1981) 有如下公式

$$\begin{aligned}
\mathrm{E}\left(x_t x_{t+r} x_s x_{s+r+v}\right) = {} & \mathrm{E}\left(x_t x_{t+r}\right)\mathrm{E}\left(x_s x_{s+r+v}\right) + \mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_{t+r} x_{s+r+v}\right) \\
& + \mathrm{E}\left(x_t x_{s+r+v}\right)\mathrm{E}\left(x_{t+r} x_s\right)
\end{aligned}$$

   通过计算说明

$$\begin{aligned}
\mathrm{Cov}\left(x_t^2, x_s x_{s+1}\right) &= 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_{s+1}\right) = 2\,\mathrm{Cov}\left(x_t, x_s\right)\mathrm{Cov}\left(x_t, x_{s+1}\right) \\
\mathrm{Cov}\left(x_t^2, x_s^2\right) &= 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_s\right) = 2\left[\mathrm{Cov}\left(x_t, x_s\right)\right]^2.
\end{aligned}$$

   参考答案

$$\begin{aligned}
\mathrm{Cov}\left(x_t^2, x_s x_{s+1}\right) &= \mathrm{E}\left(x_t^2 x_s x_{s+1}\right) - \mathrm{E}\left(x_t^2\right)\mathrm{E}\left(x_s x_{s+1}\right) \\
&= \mathrm{E}\left(x_t^2\right)\mathrm{E}\left(x_s x_{s+1}\right) + \mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_{s+1}\right) + \mathrm{E}\left(x_t x_{s+1}\right)\mathrm{E}\left(x_t x_s\right) - \mathrm{E}\left(x_t^2\right)\mathrm{E}\left(x_s x_{s+1}\right) \\
&= 2\,\mathrm{E}\left(x_t x_s\right)\mathrm{E}\left(x_t x_{s+1}\right) = 2\,\mathrm{Cov}\left(x_t, x_s\right)\mathrm{Cov}\left(x_t, x_{s+1}\right)
\end{aligned}$$

类似地可以说明

$$\mathrm{Cov}\left(x_t^2, x_s^2\right) \ = \ 2\,\mathrm{E}\,(x_t x_s)\,\mathrm{E}\,(x_t x_s) = 2\left[\mathrm{Cov}\,(x_t, x_s)\right]^2$$

(4) 计算 $\mathrm{Cov}\,(U_T, V_T)$，$\mathrm{Var}\,(V_T)$.

参考答案

$$\begin{aligned}
\mathrm{Cov}\,(U_T, V_T) \ &= \ \mathrm{Cov}\left(\frac{1}{T}\sum x_t x_{t-1}, \frac{1}{T}\sum x_{t-1}^2\right) \\
&= \ \frac{1}{T^2}\sum\sum \mathrm{Cov}\left(x_t^2, x_s x_{s+1}\right) \\
&= \ \frac{2}{T^2}\sum\sum \mathrm{Cov}\,(x_t, x_s)\,\mathrm{Cov}\,(x_t, x_{s+1}) \\
&= \ \sigma_x^4 \frac{2}{T^2}\sum\sum \phi^{|t-s|}\phi^{|t-s-1|}
\end{aligned}$$

其中，

$$\begin{aligned}
\frac{2\sigma_x^4}{T^2}\sum\sum \phi^{|t-s|}\phi^{|t-s-1|} \ &= \ \frac{2\sigma_x^4}{T^2}\left[T\phi + (T-1)(\phi+\phi^3) + (T-2)(\phi^3+\phi^5) + \cdots + (\phi^{2T-3}+\phi^{2T-1})\right] \\
&= \ \frac{2\sigma_x^4}{T^2}\left[(2T-1)\phi + (2T-3)\phi^3 + (2T-5)\phi^5 + \cdots + (2T-3)\phi^{2T-3} + \phi^{2T-1}\right] \\
&= \ \frac{2\sigma_x^4}{T^2}\frac{2T\phi}{1-\phi^2} + o\left(T^{-1}\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\,(V_T) \ &= \ \mathrm{Var}\left(\frac{1}{T}\sum x_{t-1}^2\right) \\
&= \ \frac{1}{T^2}\sum\sum \mathrm{Cov}\left(x_t^2, x_s^2\right) \\
&= \ \frac{2}{T^2}\sum\sum \left[\mathrm{Cov}\,(x_t, x_s)\right]^2 \\
&= \ \frac{2\sigma_x^4}{T^2}\sum\sum \phi^{2|t-s|}
\end{aligned}$$

其中，

$$\begin{aligned}
\frac{2\sigma_x^4}{T^2}\sum\sum \phi^{2|t-s|} \ &= \ \frac{2\sigma_x^4}{T^2}\left[T + 2(T-1)\phi + 2(T-2)\phi^2 + \cdots + 2\phi^{T-1}\right] \\
&= \ \frac{2\sigma_x^4}{T^2}\frac{T(1+\phi^2)}{1-\phi^2} + o\left(T^{-1}\right)
\end{aligned}$$

(5) 通过计算说明

$$\mathrm{E}\,(\hat{\phi}_{ols}) = \mathrm{E}\left(\frac{U_T}{V_T}\right) = \frac{\mathrm{E}\,(U_T)}{\mathrm{E}\,(V_T)} - \frac{\mathrm{Cov}\,(U_T, V_T)}{\mathrm{E}^2\,(V_T)} + \frac{\mathrm{E}\,(U_T)\,\mathrm{Var}\,(V_T)}{\mathrm{E}^3\,(V_T)} + o\left(T^{-1}\right).$$

(6) 在(1)-(5) 的基础上通过计算说明

$$\mathrm{E}\,(\hat{\phi}_{ols}) = \phi - \frac{2\phi}{T} + o\left(T^{-1}\right).$$

参考答案.

$$
\begin{aligned}
\mathrm{E}\left(\hat{\phi}_{ols}\right) = \mathrm{E}\left(\frac{U_T}{V_T}\right) &= \frac{\mathrm{E}\left(U_T\right)}{\mathrm{E}\left(V_T\right)} - \frac{\mathrm{Cov}\left(U_T, V_T\right)}{\mathrm{E}^2\left(V_T\right)} + \frac{\mathrm{E}\left(U_T\right)\mathrm{Var}\left(V_T\right)}{\mathrm{E}^3\left(V_T\right)} + o\left(T^{-1}\right) \\
&= \frac{\phi\sigma_x^2}{\sigma_x^2} - \frac{2\sigma_x^4}{T^2}\frac{2T\phi}{1-\phi^2}\frac{1}{\sigma_x^4} + \frac{\phi\sigma_x^2 2\sigma_x^4 T(1+\phi^2)}{T^2(1-\phi^2)\sigma_x^6} + o\left(T^{-1}\right) \\
&= \phi - \frac{4\phi}{T(1-\phi^2)} + \frac{2\phi(1+\phi^2)}{T(1-\phi^2)} + o\left(T^{-1}\right) \\
&= \phi + \frac{2\phi\left(1+\phi^2-2\right)}{T(1-\phi^2)} + o\left(T^{-1}\right) \\
&= \phi - \frac{2\phi}{T} + o\left(T^{-1}\right)
\end{aligned}
$$